Weighted Lattice Paths

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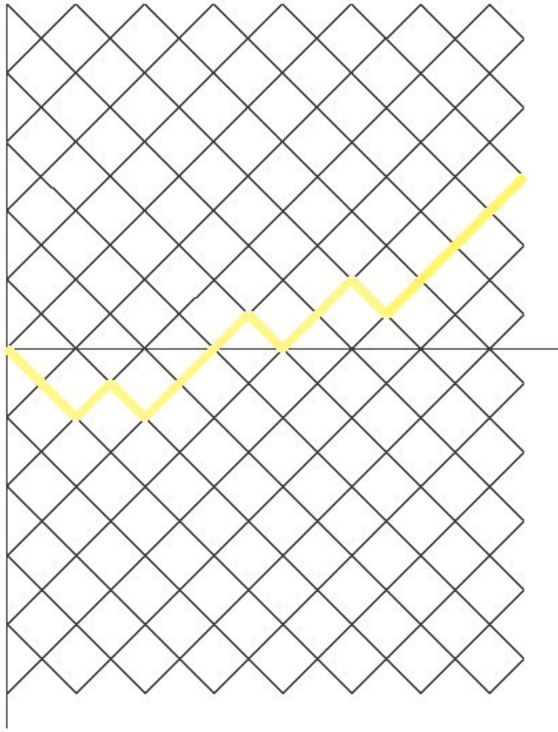
Binomial Paths and the constant term method



$$W_{t,y} = \text{number of t-step paths ending at y}$$
$$= \binom{t}{\frac{1}{2}(t-y)}$$
$$= CT[(z+z^{-1})^t z^y]$$

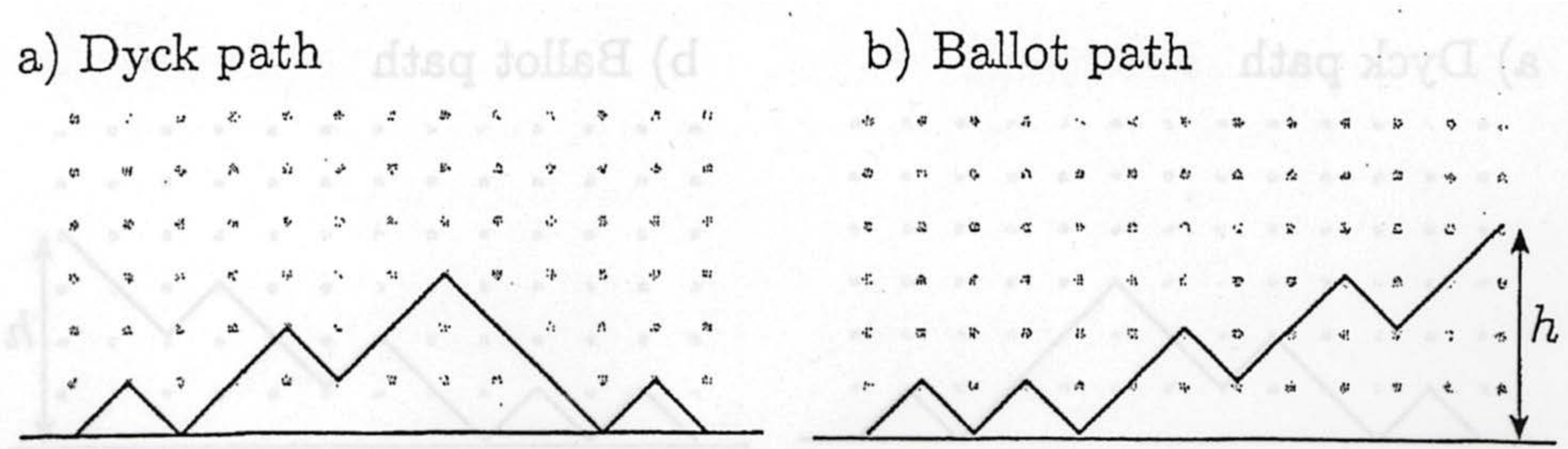
$CT[f(z)]$ means select the term independent of z in the laurent expansion of $f(z)$.

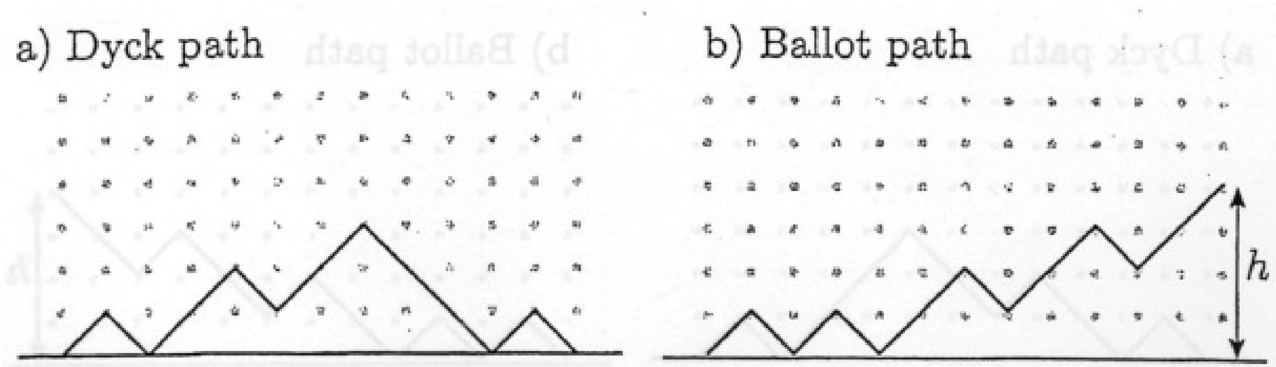
Method due to P A MacMahon ,*Combinatory Analysis Vol. 2 1916*

Dyck and Ballot paths

- A Dyck path is a lattice path which starts and ends on the x-axis, avoiding the region below.

- A Ballot path is a lattice path which starts on the x-axis, avoids the region below and ends at height h.



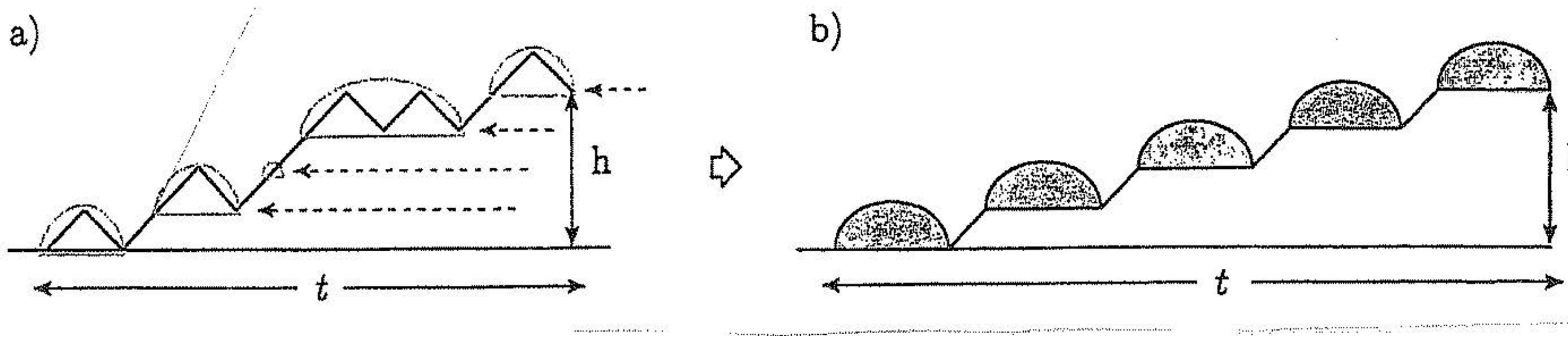


Constant term formulae

$$
\begin{aligned}
B_{t,h} &= \text{number of t-step Ballot paths ending at height h} \\
&= W_{t,h} - W_{t,h+2} \\
&= CT[\Lambda^t z^h (1 - z^2)] \qquad \text{where} \qquad \Lambda = z + z^{-1} \\
C_r &= \text{number of Dyck paths making } 2r \text{ steps} = B_{2r,0} \\
&= CT[\Lambda^{2r}(1 - z^2)] \\
&= \frac{1}{r+1}\binom{2r}{r}
\end{aligned}
$$

Catalan numbers: 1 1 2 5 14 42 132

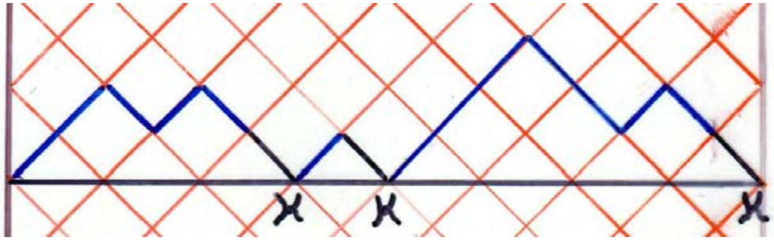
Diagrammatic representation of a Ballot Path

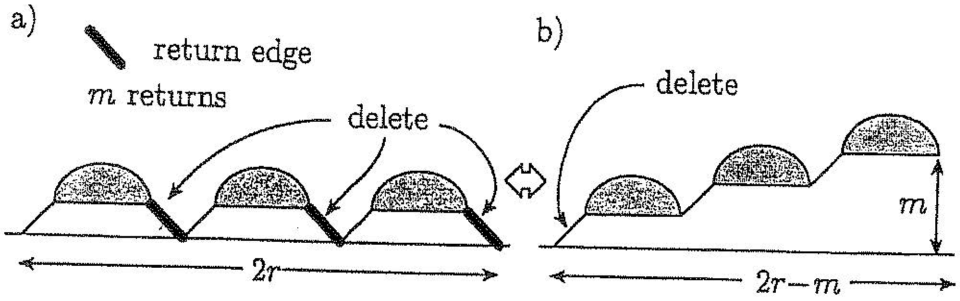


Represents any Dyck path including a single site

Polymer adsorption on a surface

- The diagram represents a polymer chain attached to a surface at both ends.





The **partition function** of the polymer or **return polynomial** is

$$\hat{Z}_r(\kappa) = \sum_{m=0}^{r} C_{r,m}\kappa^m$$

where $C_{r,m}$ is the number of Dyck paths of length $2r$ having m returns

From the diagram $\qquad C_{r,m} = B_{2r-m-1,m-1}$

$$\hat{Z}_r(\kappa) = \sum_{m=0}^{r} B_{2r-m-1,m-1}\kappa^m = \sum_{m=0}^{\infty} CT[\Lambda^{2r-1}(1-z^2)z^{-1}(z\kappa/\Lambda)^m]$$

$$= CT[\frac{\Lambda^{2r-1}(1-z^2)z^{-1}}{1-z\kappa/\Lambda}]$$

The omega variable and the absorption transition

As κ increases there comes a point when the polymer sticks to the surface. The sticking point may be deduced from the asymptotic form of the partition function as $r \to \infty$. Using symmetry of $CT[]$ under interchange of z and z^{-1}

$$\hat{Z}_r(\kappa) = CT[\frac{\Lambda^{2r-1}(1-z^2)z^{-1}}{1-z\kappa/\Lambda}] = CT[\frac{\Lambda^{2r}(1-z^2)}{1-\omega\Lambda^2}]$$

where $\omega = (\kappa - 1)/\kappa^2$

Expanding the factor $1/(1-\omega\Lambda^2)$ gives an infinite series in powers of ω which turns out to be only valid for $\kappa \leq 2$. Instead we use the CT formula to obtain a recurrence relation valid for all κ. Thus noting that

$$\frac{\omega\Lambda^{2r}}{1-\omega\Lambda^2} = -\Lambda^{2r-2} + \frac{\Lambda^{2r-2}}{1-\omega\Lambda^2}$$

gives

$$\omega\hat{Z}_{2r}(\kappa) = -C_{r-1} + \hat{Z}_{2r-2}(\kappa).$$

Solving subject to $\hat{Z}_2(\kappa) = \kappa^2$ gives

$$\hat{Z}_{2r}(\kappa) = \omega^{-r}(\kappa - \sum_{j=0}^{r-1} C_j\omega^j)$$

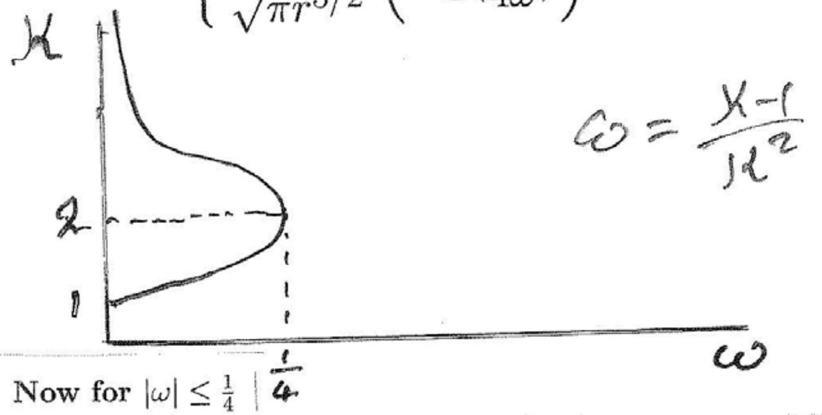
which on substituting for ω in terms of κ must give a polynomial.

$Z_{2r} = Z_r$

Proposition 1.2. *For $r \to \infty$*

$$\hat{Z}_{2r}(\kappa) \sim \begin{cases} \dfrac{\kappa(\kappa-2)}{\kappa-1}\omega^{-r} & \kappa > 2 \\ 2\dfrac{4^r}{\sqrt{\pi r}} & \kappa = 2 \\ \dfrac{4^r}{\sqrt{\pi} r^{3/2}}\left(\log(\frac{1}{4\omega})\right)^{-1} & \kappa < 2 \end{cases}$$

$$\log\left(\frac{1}{4\omega}\right) \sim \frac{1}{4}(\kappa-2)^2$$



Proof. Now for $|\omega| \le \frac{1}{4}$

$$\sum_{s=0}^{\infty} C_s \omega^s = \frac{1-\sqrt{1-4\omega}}{2\omega} = \begin{cases} \kappa & \text{for } \kappa \le 2 \\ \kappa/(\kappa-1) & \text{for } \kappa > 2 \end{cases}$$

and hence

$$\hat{Z}_{2r}(\kappa) = \frac{\kappa(\kappa-2)}{\kappa-1}\omega^{-r}\theta(\kappa-2) + \sum_{s=r}^{\infty} C_s \omega^{s-r}.$$

The Catalan numbers have asymptotic form

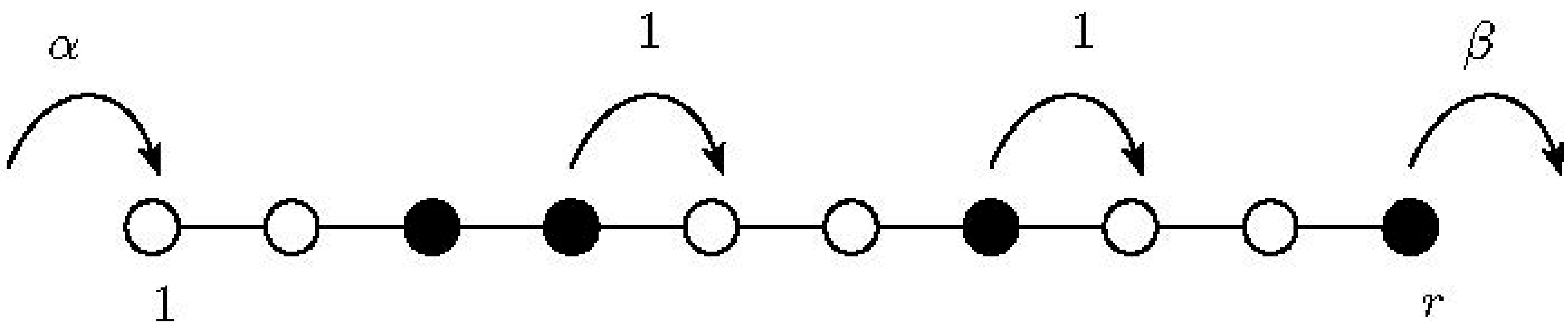
$$C_s \sim \frac{4^s}{\pi^{\frac{1}{2}} s^{\frac{3}{2}}} \qquad \text{as } s \to \infty\,,$$

and replacing the sum by an integral gives

$$\hat{Z}_{2r}(\kappa) \sim \frac{\kappa(\kappa-2)}{\kappa-1}\omega^{-r}\theta(\kappa-2) + \frac{4^r}{\sqrt{\pi r}}\chi(r\log(\frac{1}{4\omega})).$$

where

$$\chi(y) = \int_1^{\infty} \frac{e^{-y(u-1)}}{u^{3/2}} du \sim \frac{1}{y} - \frac{3}{2y^2} + O(\frac{1}{y^3}) \qquad \text{as } y \to \infty.$$



The TASEP

α = hop on rate

β = hop off rate

The unnormalised steady state probabilities are denoted $f_N(\tau_1, \tau_2, \ldots, \tau_N)$ where $\tau_i = 1$ if site i is occupied and 0 otherwise. The normalising factor is

$$Z_N = \sum_{\tau_1, \tau_2, \ldots, \tau_N} f_N(\tau_1, \tau_2, \ldots, \tau_N)$$

Early ASEP references

B. Derrida, E. Domany and D. Mukamel, "An exact solution of a one-dimensional asymmetric exclusion model with open boundaries", J. Stat. Phys. **69** 667-87 (1992)

G. Schütz and E. Domany, "Phase transitions in an exactly soluble one dimensional exclusion process", J. Stat. Phys. **72** 277-96 (1993)

B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, "Exact solution of a 1D asymmetric exclusion model using a matrix formulation", J. Phys. A: Math. Gen. **26** 1493-517 (1993)

$$f_1(0) = \bar{\alpha}, \qquad f_1(1) = \bar{\beta}$$

$$\begin{aligned}
f_N(0,0,\ldots,0) &= \bar{\alpha} f_{N-1}(0,0,\ldots,0) \\
f_N(\tau_1,\tau_2,\ldots,\tau_{N-1},1) &= \bar{\beta} f_{N-1}(\tau_1,\tau_2,\ldots,\tau_{N-1}) \\
f_N(\tau_1,\tau_2,\ldots,\tau_{i-1},1,0,\ldots,0) &= \sum_{\tau_i=0}^{1} f_{N-1}(\tau_1,\tau_2,\ldots,\tau_{i-1},\tau_i,0,\ldots,0)
\end{aligned}$$

Define

$$Y_{N,k} \equiv \sum_{\tau_1,\ldots,\tau_{k-1}} f_N(\tau_1,\ldots,\tau_{k-1},0,\ldots,0)$$

so $Z_N = Y_{N,N+1}$. For $2 \leq k \leq N+1$, $Y_{n,k}$ satisfies the recurrence relation

$$Y_{N,k} = Y_{N,k-1} + Y_{N-1,k}$$

with boundary conditions $\quad Y_{N,1} = \bar{\alpha} Y_{N-1,1} \quad$ and $\quad Y_{N-1,N+1} = \bar{\beta} Y_{N-1,N}$

These equations were solved by Derrida et al (1992 J Stat Phys **69** 667-87) in the case $\alpha = \beta = 1$ using generating functions and extension to arbitrary α and β was done by Schütz and Domany

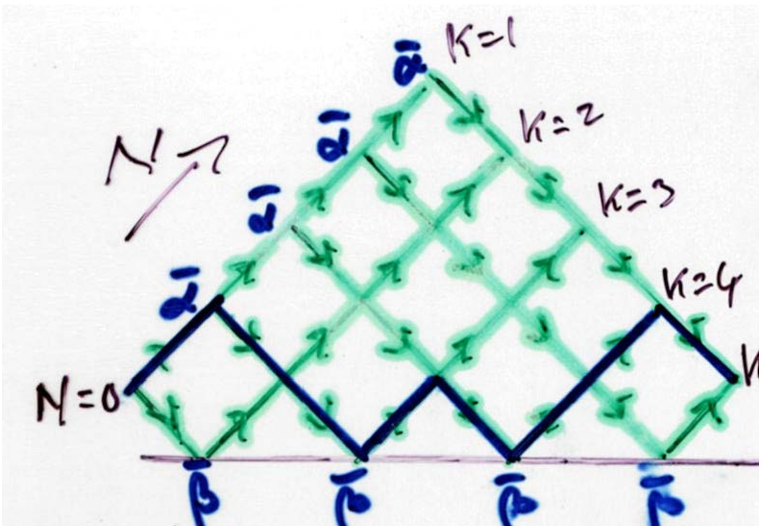
The above equations are those satisfied by the number of directed walks $Y_{N,k}$ from $(0,1)$ to (N,k) weighted as on the diagram below. This is related to the return polynomial $R_t(h,\kappa)$ for $t-$step paths ending at height h which, following the derivation of $\hat{Z}_r(\kappa)$, is given by

$$R_t(h,\kappa) = \sum_{m=0}^{\frac{1}{2}(t-h)} B_{t-m-1,m+h-1}\kappa^m.$$

The coefficient of $\bar{\alpha}^h$ in Z_N comes from paths which start by making h steps along the upper boundary which is followed by a Ballot path with $t = 2N - h$ steps starting at height h and ending at height zero.

$$Z_N = \sum_{h=0}^{N} R_{2N-h}(h;\bar{\beta})\bar{\alpha}^h = \sum_{h=0}^{N} \bar{\alpha}^h \sum_{j=0}^{N-h} B_{2N-h-j-1,j+h-1}\bar{\beta}^j$$

Lattice path, a term of Z4



ASEP PHASES

Z_N may be rewritten

DEHP equation (39)

$$Z_N = \sum_{m=0}^{N} B_{2N-m-1,m-1} \sum_{j=0}^{m} \bar{\alpha}^{m-j}\bar{\beta}^{j} = \sum_{m=0}^{N} B_{2N-m-1,m-1} \frac{\bar{\alpha}^{m+1} - \bar{\beta}^{m+1}}{\bar{\alpha} - \bar{\beta}}$$

$$= \frac{\hat{Z}_N(\bar{\alpha}) - \hat{Z}_N(\bar{\beta})}{\bar{\alpha} - \bar{\beta}}$$

The Dyck path contact polynomial $\hat{Z}_N(\bar{\alpha})$ has asymptotic form

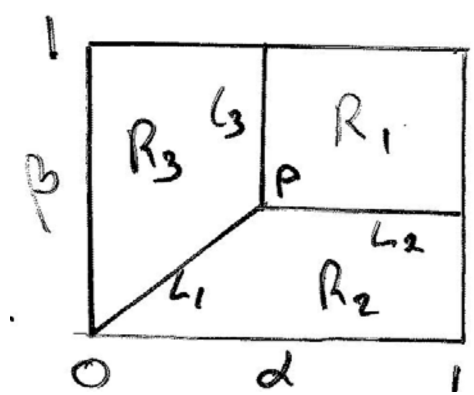
$$\hat{Z}_N(\bar{\alpha}) \sim \begin{cases} f_<(\alpha) \equiv \frac{1-2\alpha}{\omega_\alpha^{N+1}} & \alpha < \frac{1}{2} \\ f_= \equiv \frac{2}{\sqrt{\pi}} \frac{4^N}{N^{\frac{1}{2}}} & \alpha = \frac{1}{2} \\ f_>(\alpha) \equiv \frac{4^N}{\sqrt{\pi} N^{\frac{3}{2}}} (1 - 4\omega_\alpha)^{-1} & \alpha > \frac{1}{2} \end{cases}$$

R_1 = maximal current phase

R_2= High density phase

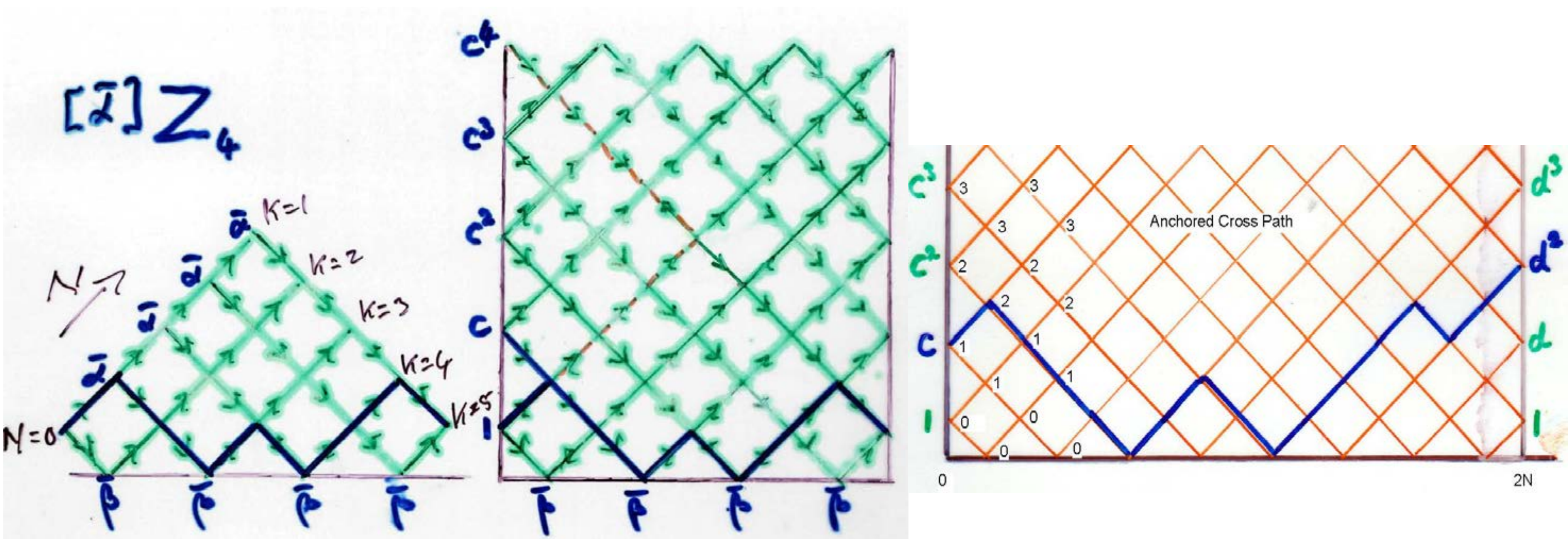
R_3 = Low density phase

where $\omega_\alpha = \alpha(1-\alpha)$.



$R_3 : \frac{f_<(\alpha)}{\bar{\alpha}-\bar{\beta}}$	$L_3 : \frac{f_=}{2-\bar{\beta}}$	$R_1 : \frac{f_>(\alpha)-f_>(\beta)}{\bar{\alpha}-\bar{\beta}}$
	$P : 4^r$	$L_2 : \frac{f_=}{2-\bar{\alpha}}$
$L_1 : -\alpha^2 f'_<(\alpha) \sim \frac{r(1-2\alpha)^2}{\omega_\alpha^{r+2}}$		$R_2 : \frac{f_<(\beta)}{\bar{\beta}-\bar{\alpha}}$

Transfer matrix



$$D_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

$$\bar{\alpha} = 1 + c$$

$$\bar{\beta} = 1 + d$$

$$\kappa^2 = 1 - cd$$

$$\langle W_2| = \kappa\,(1, c, c^2, c^3, \ldots) \qquad |V_2\rangle = \kappa\,(1, d, d^2, d^3, \ldots)^T$$

DEHP

$$Z_N = < W_2|(D_2E_2)^N|V_2 >= \sum_{i,j=0}^{\infty} B_{2N+1,2i+2j+1} c^i d^j$$

Compact percolation
Compact cluster with bounding vesicle

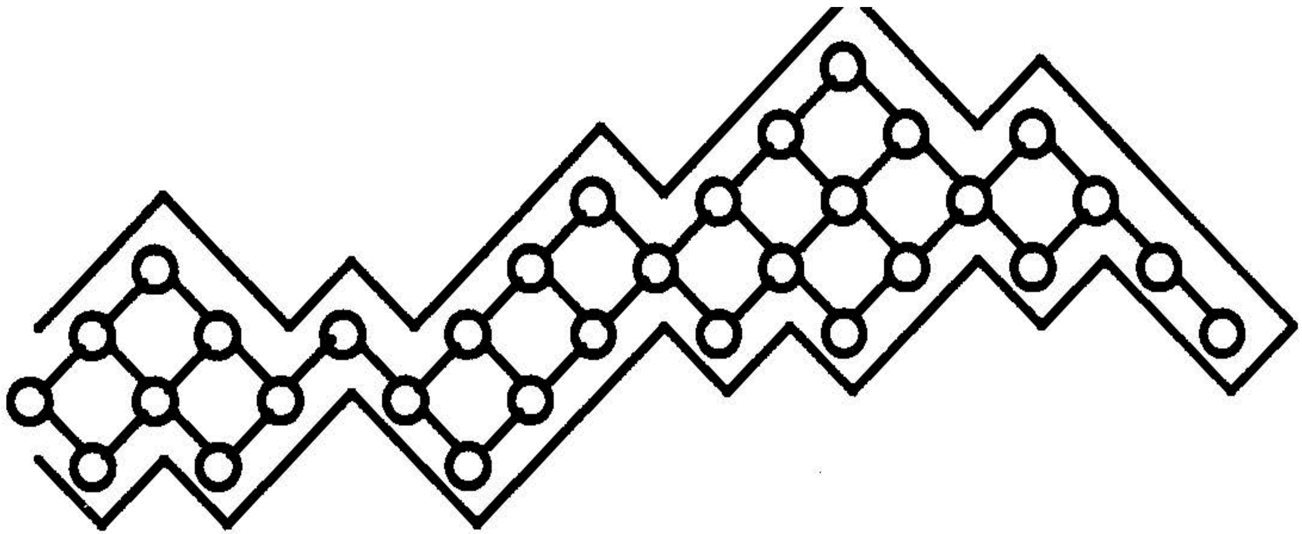


Figure 1. A directed compact cluster with 19 growth stages, length 20 and size 32 together

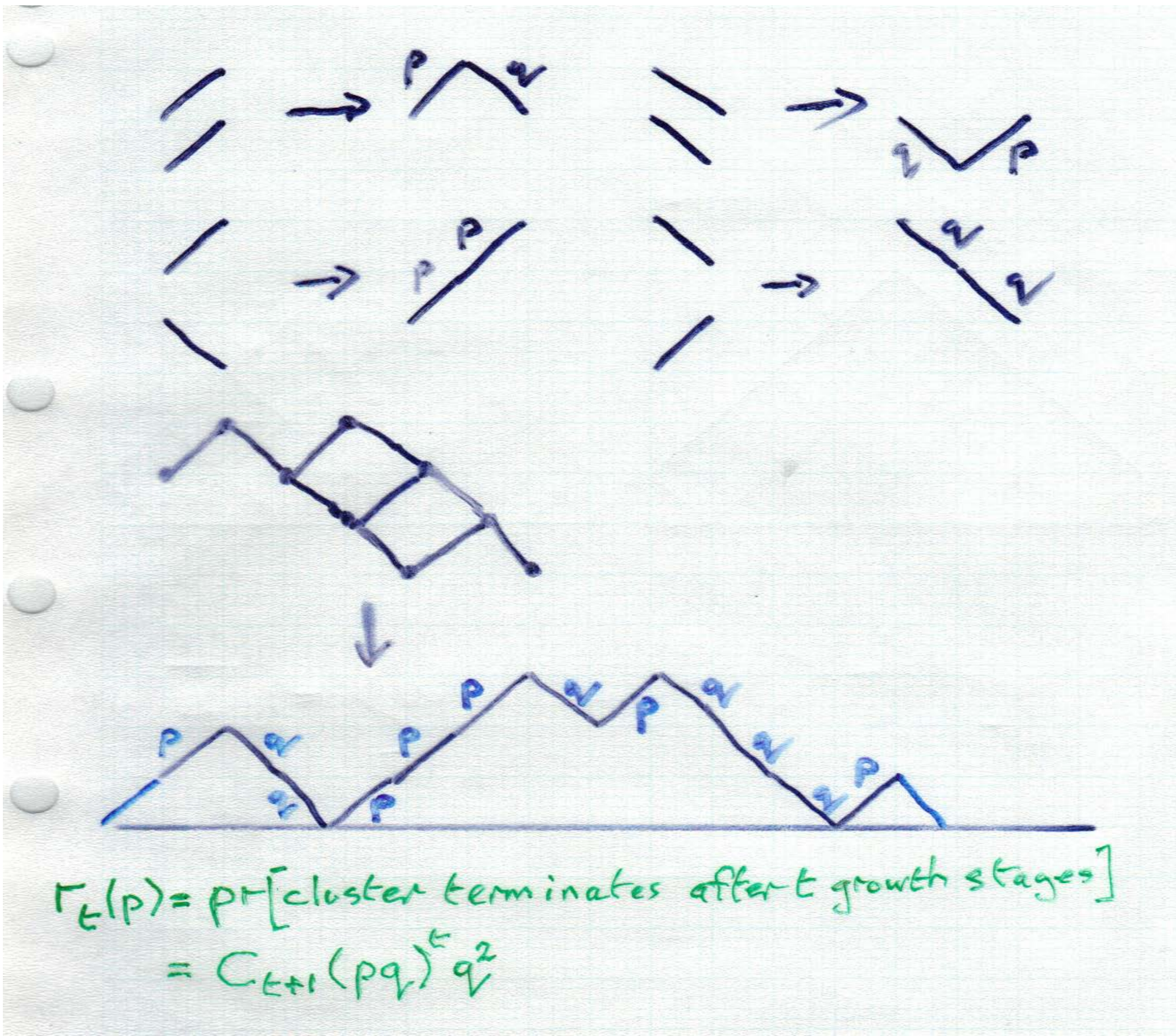
Top edge moves up with probability p and down with probability q = 1-p

Bottom edge moves down with probability p and up with probability q

Cluster terminates with probability q^2

Bijection: Compact clusters to Dyck paths

Domany and Kinzel 1984



Compact percolation: Critical exponents

$$r_t(p) \cong e^{-t/\xi_{\Uparrow}(p)} / \pi^{1/2} t^{3/2}$$ (Domany and Kinzel 1984)

Parallel connectedness length (exponent $\nu_{\Uparrow}$)

$$\xi_{\Uparrow}(p) = (1 - 2p)^{-2}$$ $\nu_{\Uparrow} = 2$

Percolation probability (exponent β)

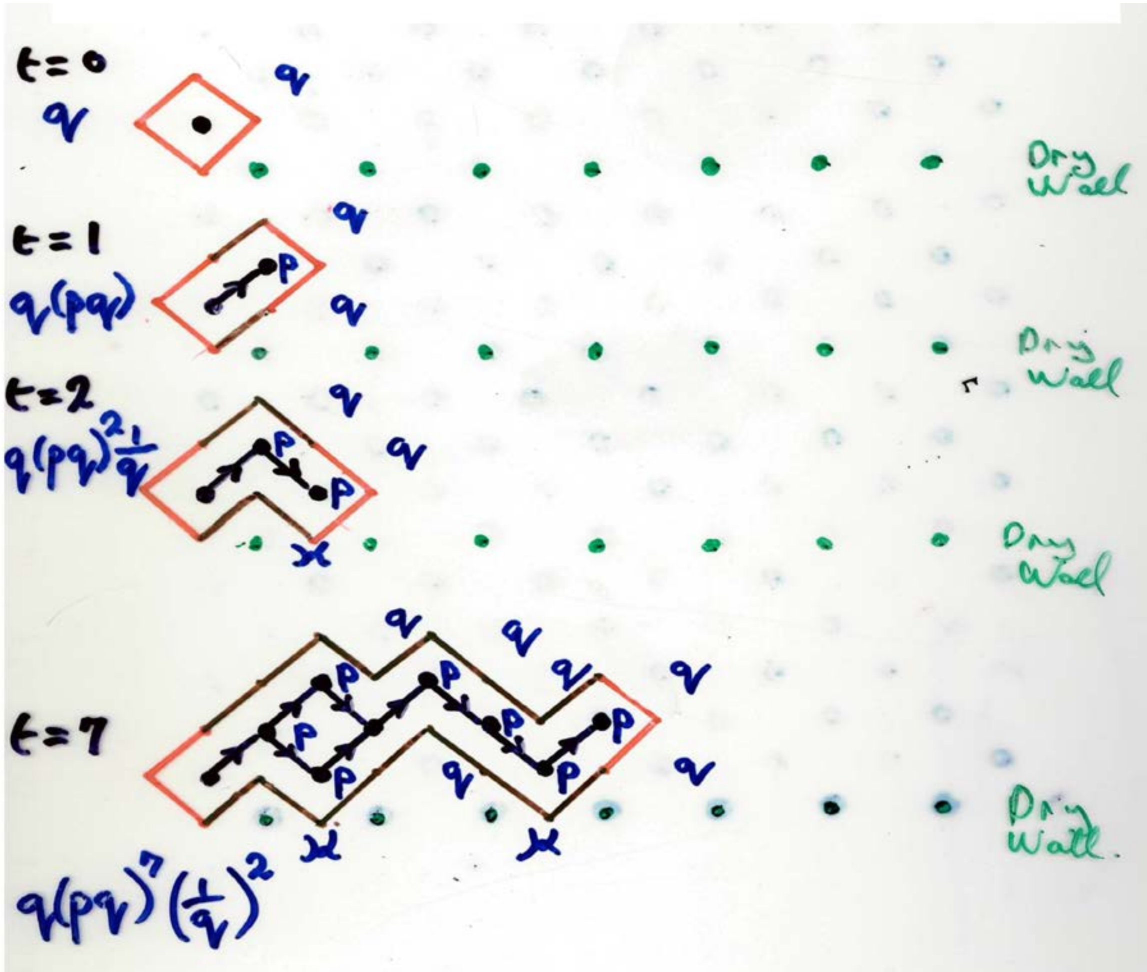
$$P(p) = 1 - \sum_{t=0}^{\infty} r_t(p) = \begin{cases} 0 & p < p_c = 1/2 \\ (2p-1)/p^2 & p \geq p_c \end{cases} \quad (\beta = 1)$$

Mean cluster length (exponent τ)

$$L(p) = \sum_{t=0}^{\infty} (t+1) r_t(p) = |1 - 2p|^{-1} \quad (\tau = 1)$$

Scaling $\tau + \beta = \nu_{\Uparrow}$

Compact clusters and vesicles near a wall



Note: factor kappa =1/q for each return to the wall and factor pq for each of t steps

Compact percolation and vesicles near a wall

The vesicles can end anywhere above the surface.

Joint work with Richard Brak.

The **vesicle grand partition function** is

$$Z(u,\kappa) = \sum_{t=0}^{\infty} V_t(\kappa) u^t$$

where $V_t(\kappa)$ is the partition function for vesicles constructed from walks of length t the Boltzmann weight having a factor κ for each contact with the wall. (Except the first)

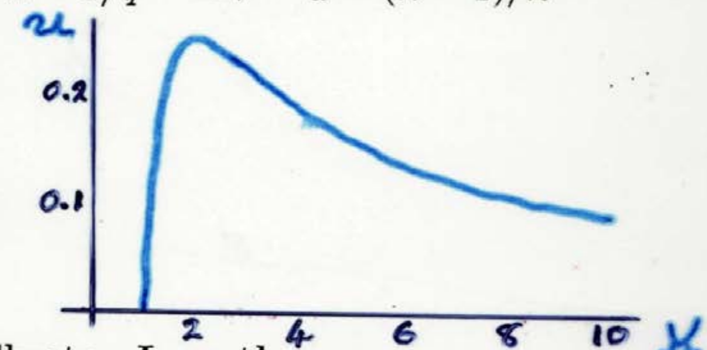
Connection with Percolation probability

$$P(p) = 1 - qZ(pq, 1/q)$$

The percolation line.

$$u = pq, \kappa = 1/q \quad \Longrightarrow \quad u = (\kappa - 1)/\kappa^2$$

Adsorption occurs at $\kappa_c = 2$. Corresponds to $p_c = \frac{1}{2}$.



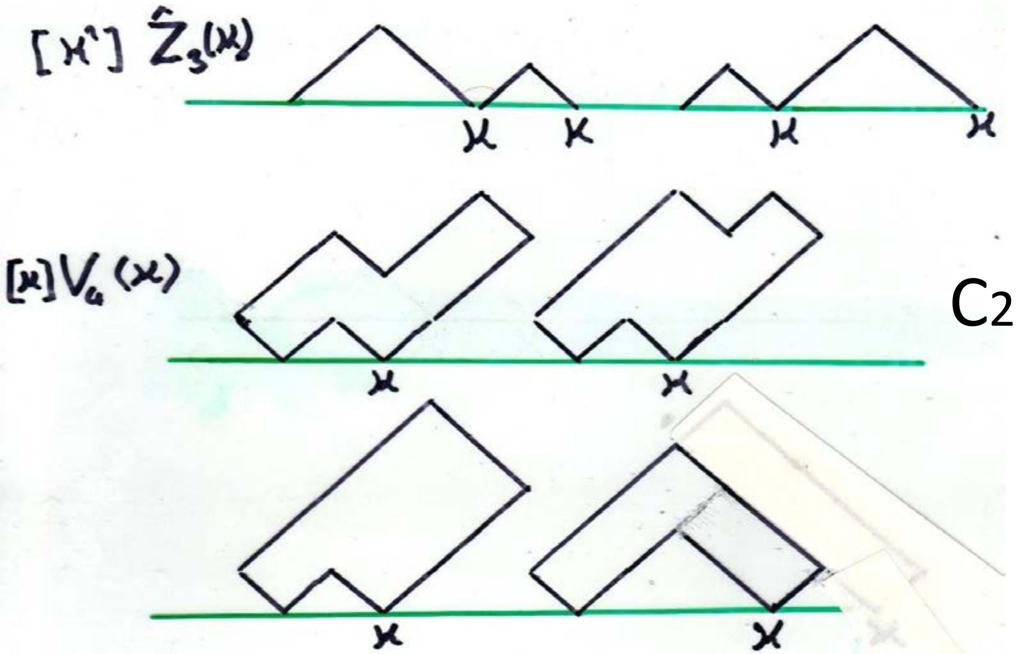
The Mean Cluster Length

$$L(p) = q \frac{\partial}{\partial u}(uZ(u,\kappa))\Big|_{\kappa=1/q, u=pq}$$

The Mean Number of Wall Contacts

$$N(p) = q \frac{\partial}{\partial \kappa}(\kappa Z(u,\kappa))\Big|_{\kappa=1/q, u=pq}$$

Relation between vesicles and single chains



$$V_{2r}(\kappa) = C_r \hat{Z}_{r+1}(\kappa)/\kappa$$

C2=2

$$V_{2r+1}(\kappa) = C_{r+1} \hat{Z}_{r+1}(\kappa)/\kappa$$

$$Z(u,\kappa) = \sum_{t=0}^{\infty} V_t(\kappa) u^t = \sum_{r=0}^{\infty} (C_r + uC_{r+1}) \hat{Z}_{r+1}(\kappa)/\kappa$$

$$= Z^+(u,\kappa)\theta(\kappa-2) = \frac{1}{\kappa-1} \sum_{r=0}^{\infty} u^{2r}(C_r + uC_{r+1}) \sum_{s=r+1}^{\infty} C_s \omega^{s-r}$$

where $\omega = (\kappa-1)/\kappa^2$ and

$$Z^+(u,\kappa) = \frac{\kappa(\kappa-2)}{(\kappa-1)^2} \left[1 + \left(1 + \frac{\omega}{u}\right) \left(\frac{\omega}{2u^2} - 1 - \frac{\sqrt{\omega(\omega - 4u^2)}}{2u^2} \right) \right]$$

Percolation Probability

$$P(p) = 1 - (1-p)Z(pq, 1/q)$$

where

$$Z(pq,1/q) = \frac{(2-p)(2p-1)}{p^3}\theta(p-\frac{1}{2}) + \frac{q}{p}\left[\left(\sum_{r=0}^{\infty} C_r u^r\right)^2 - \sum_{r=0}^{\infty} C_r u^r\right]$$

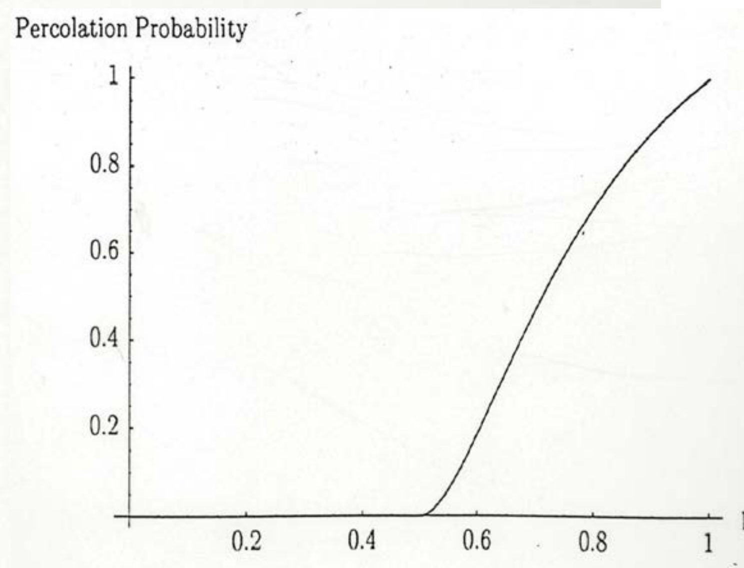
u=pq

and using

$$\sum_{r=0}^{\infty} C_r (pq)^r = \begin{cases} 1/q & p \le p_c \\ 1/p & p > p_c \end{cases}$$

rederives the result of J. C. Lin

$$P(p) = \begin{cases} 0 & p \le p_c \\ \frac{(2p-1)^2}{p^3} & p > p_c. \end{cases}$$



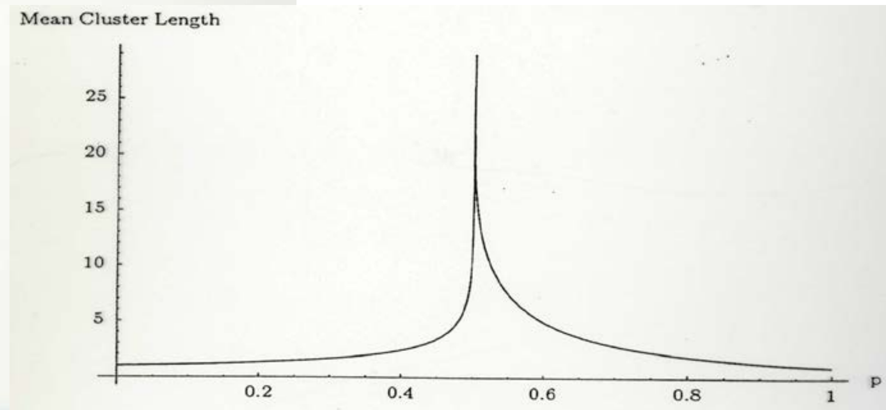
Mean Length of Compact Clusters

$$L(p) = q\frac{\partial}{\partial u}\left(uZ(u,\kappa)\right)\Big|_{\kappa=1/q,u=pq}$$

$$= \theta(p-p_c)\frac{q(3-2p)}{p^3} + \frac{q^2}{p}\sum_{k=1}^{\infty}(a_k u^k + b_k u^{k+1})$$

where

$$a_k = \sum_{r=0}^{\lfloor\frac{1}{2}(k-1)\rfloor}(2r+1)C_r C_{k-r} \quad \text{and} \quad b_k = \sum_{r=0}^{\lfloor\frac{1}{2}(k-1)\rfloor}(2r+2)C_{r+1}C_{k-r}.$$



Using Zeilberger's algorithm

$$a_{2s+1} = \binom{2s+1}{s}\binom{2s+2}{s+1} - \frac{1}{2s+3}\binom{4s+3}{2s+1}$$

$$a_{2s+2} = \frac{1}{2}\binom{2s+3}{s+1}^2 - \frac{1}{2s+4}\binom{4s+5}{2s+2}$$

with similar expressions for b_{2s+1} and b_{2s+2}.

Using Mathematica

$$L(p) = \theta(p-p_c)\frac{q(3-2p)}{p^3} + \frac{1}{8p^3}\Big\{-5+4u+6\sqrt{1-4u} - [8E(16u^2) - 2(3-4u)(1+4u)K(16u^2)]/\pi\Big\}.$$

γ=0

u=p(1-p)

Asymptotic form near p_c

$$L(p) \cong B\log|1-2p| + C^{\pm}$$

where

$$B = -\frac{8}{\pi} \quad \text{and} \quad C^{\pm} = \frac{4(3\log 2 - 2)}{\pi} \mp 4$$

The Mean Number of Wall Contacts

$$N(p) = q\frac{\partial}{\partial\kappa}\left(\kappa Z(u,\kappa)\right)\Big|_{\kappa=1/q,u=pq}$$

$$= \theta(p-p_c)\frac{q(1-2q^3)}{p^4} - \frac{q}{p}(1-P(p)) + N^*(p)$$

where

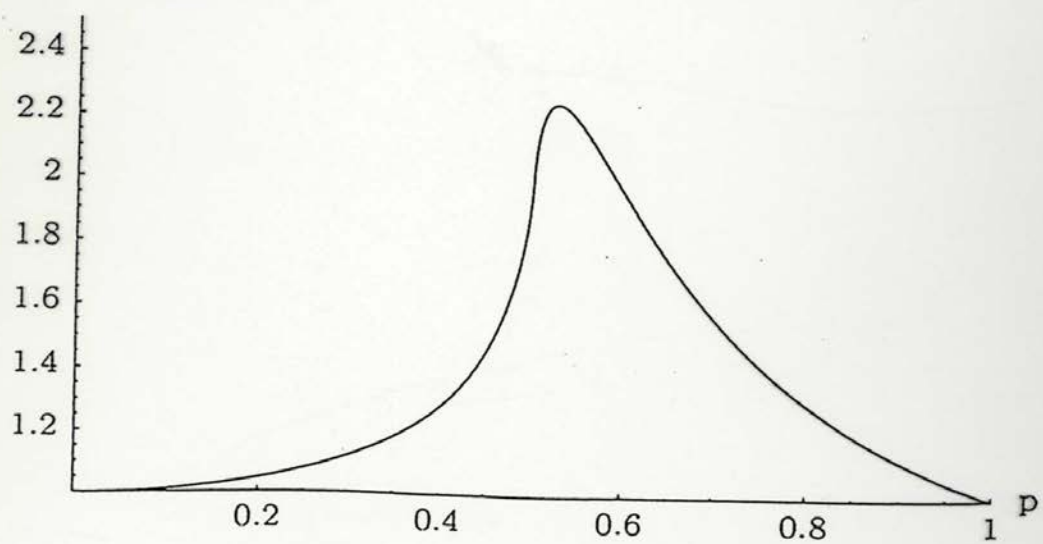
$$N^*(p) = \frac{(1-2p)}{8p^4}\Big\{1-4u-2(1-2u)\sqrt{1-4u}+$$

$$\frac{4u(1+2u)}{\sqrt{1-4u}} + \frac{8E(16u^2)}{\pi} - \frac{2(3-4u)(1+4u)K(16u^2)}{\pi}\Big\}.$$

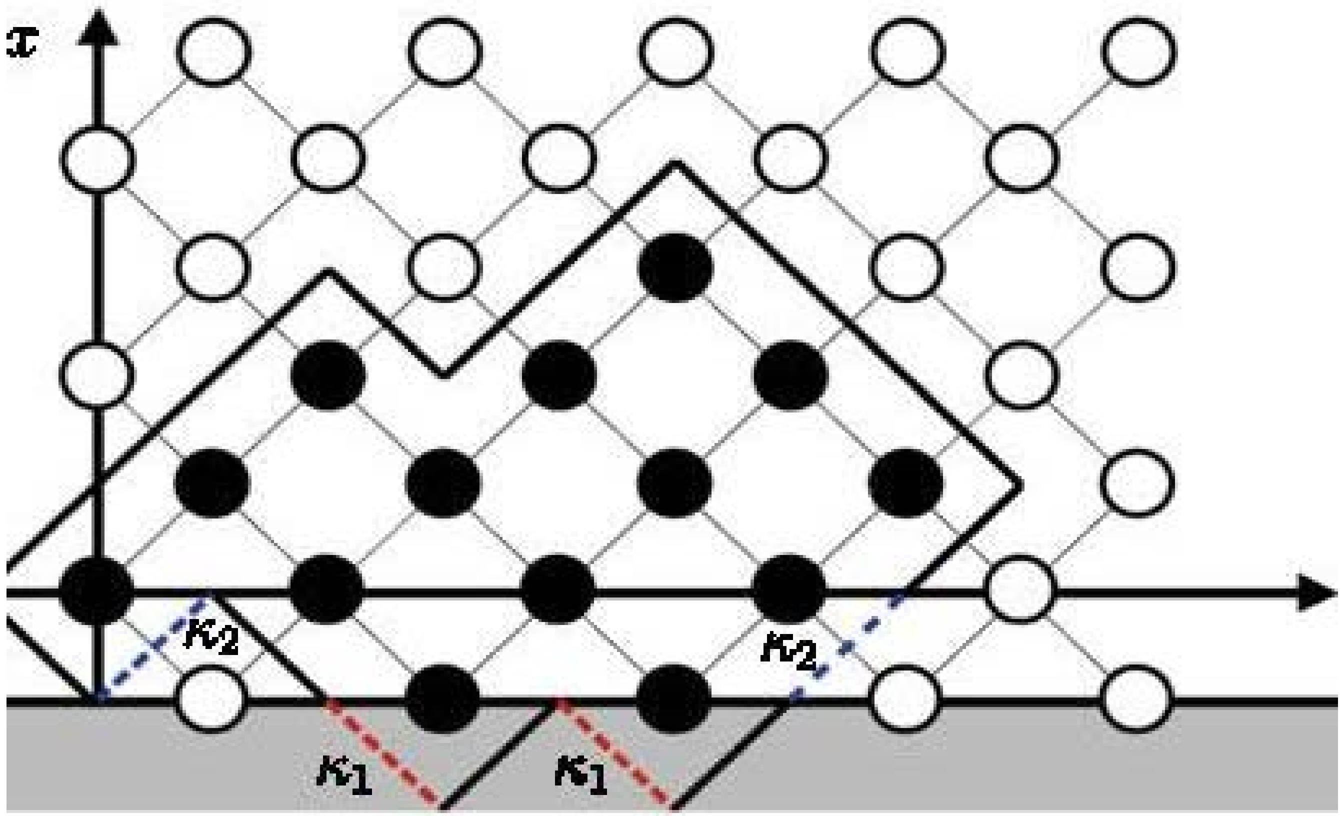
Asymptotic form near p_c

$$N(p) \cong 2 + \frac{16}{\pi}(1-2p)\log|1-2p|.$$

Mean Wall Contact Number



Compact percolation with a damp wall
with A Owczarek, R Brak, H Lonsdale and A Rechnitzer



- Wall sites wet with probability p_w

Vesicles with a compound sticky wall

$Z_{2r}(\kappa_1,\kappa_2) \equiv$ the partition function for Dyck paths with two surface weights

$$= \kappa_2 CT\left[\frac{\Lambda^{2r}(1-z^4)}{1-(\kappa_1+\kappa_2-2)z^2+(1-\kappa_2)z^4}\right]$$

$$= \kappa_2 CT\left[\frac{\Lambda^{2r}(1-z^4)}{(1-cz^2)(1-dz^2)}\right]$$

$$= \frac{\kappa_2}{c-d}\left(\frac{c}{1+c}\hat{Z}_{r+1}(\kappa_1)-\frac{d}{1+d}\hat{Z}^{r+1}(\kappa_2)\right)$$

where $\kappa_1=(1+c)(1+d), \kappa_2=1-cd$.

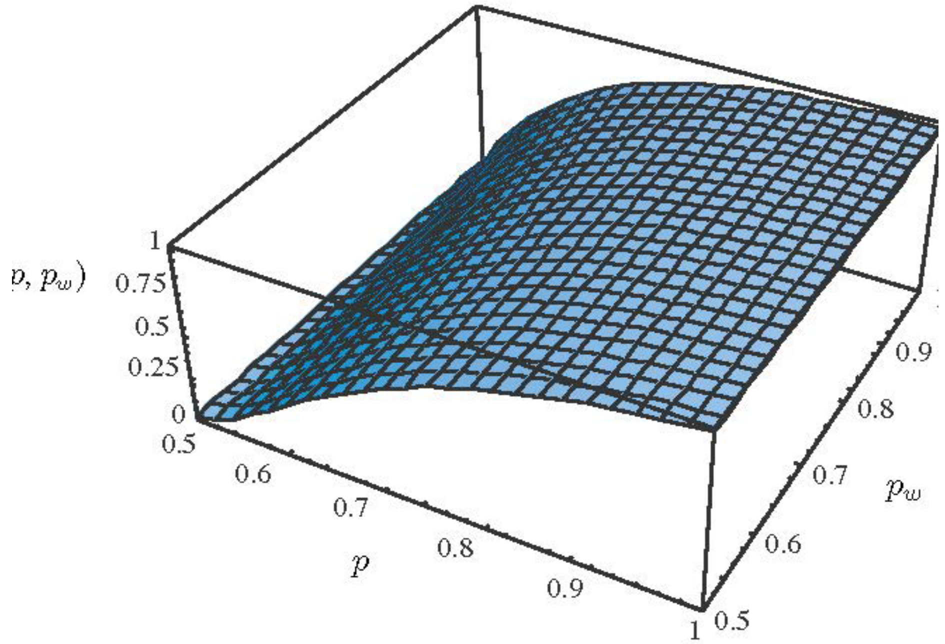
$V_t(\kappa_1,\kappa_2)$ is the partition function for vesicles with two surface weights terminating anywhere above the wall. With $\omega_c=c/(1+c)^2, \omega_d=d/(1+d)^2$

$$Z(u,\kappa_1,\kappa_2) \equiv \sum_{t=0}^{\infty} V_t(\kappa_1,\kappa_2)u^t$$

$$= \frac{1}{\kappa_2}\sum_{r=0}^{\infty} C_{r+1}[Z_{2r}(\kappa_1,\kappa_2)u^{2r+2}+Z_{2r+2}(\kappa_1,\kappa_2)u^{2r+3}] - \frac{\omega_c\omega_d u^2}{u^2-\omega_c\omega_d}$$

$$+ \frac{\kappa_2-1}{\kappa_2\kappa_1^2(u^2-\omega_c\omega_d)}\sum_{r=1}^{\infty}[C_r Z_{2r+2}(\kappa_1,\kappa_2)-C_{r+1}Z_{2r}(\kappa_1,\kappa_2)]u^{2r+2}$$

Damp wall. Percolation probability



The damp wall percolation probability is obtained by setting

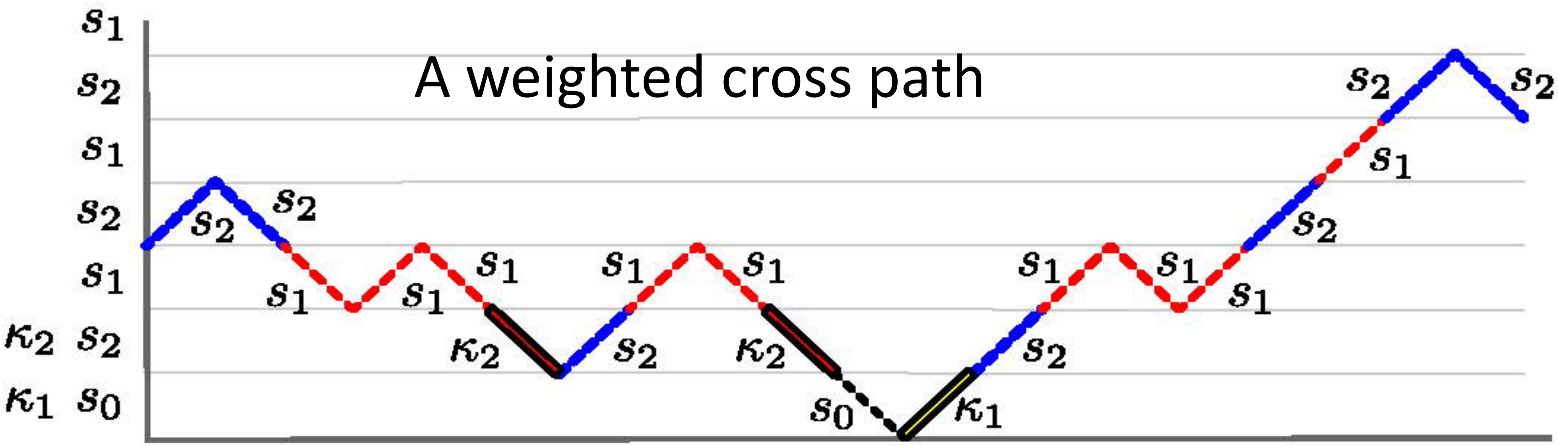
$$\kappa_1 = p_w/(pq), \kappa_2 = q_w/q \qquad \text{and} \qquad u = pq$$

$$c = \frac{p}{q},\ d = \frac{p_w - p)}{p},\ \omega_c = pq \qquad \text{and} \qquad \omega_d = \frac{p(p_w - p)}{p_w^2}$$

For $p \leq p_c = \frac{1}{2}, P(p, p_w) = 0$ and for $p > p_c$

$$P(p, p_w) = 1 - \frac{1}{p^2} Z(pq, \frac{p_w}{pq}, \frac{q_w}{q}) = \frac{(2p-1)^2}{p^2(p - p_w + pp_w)}$$

Notice: The damp wall exponent is $\beta = 2$, the same as the dry wall, except $\beta = 1$ for the wet wall $p_w = 1$.



$Z_{2r}(2k+1|2j+1)$ is a sum of the illustrated weights over paths starting at height $2j+1$ and ending at height $2k+1$. For $k \geq 1$

$$Z_{2r}(2k+1|2j+1) = s_1 s_2\big(Z_{2r-2}(2k-1|2j+1) + Z_{2r-2}(2k+3|2j+1)\big) + (s_1^2+s_2^2)Z_{2r-2}(2k+1|2j+1)$$

and

$$Z_{2r}(1|1) = s_1\kappa_2 Z_{2r-2}(3|1) + (s_0\kappa_1 + s_2\kappa_2)Z_{2r-2}(1|1).$$

Considering walks of zero length leads to the initial condition $Z_0(2k+1|2j+1) = \delta_{jk}$.

Solving the above equations gives

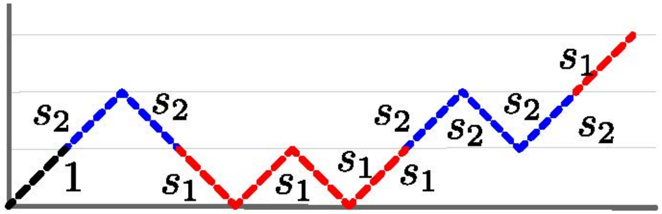
$$\hat{Z}_r(\kappa_1,\kappa_2) = Z_{2r}(1|1) = \mathrm{CT}\left[\frac{(\lambda\bar{\lambda})^r(1-z^4)}{1-(s_1s_2)^{-1}(s_0\kappa_1+s_2\kappa_2-s_1^2-s_2^2)z^2+(1-\frac{\kappa_2}{s_2})z^4}\right]$$

where $\lambda = s_1\bar{z} + s_2 z$, $\bar{\lambda} = s_1 z + s_2\bar{z}$. Factorizing the denominator gives

$$H_{2r}^{=}(c,d) = Z_{2r}(1|1) = \mathrm{CT}\left[\frac{(\lambda\bar{\lambda})^r(1-z^4)}{(1-cz^2)(1-dz^2)}\right] = \mathrm{CT}\left[(\lambda\bar{\lambda})^r\frac{(1-z^4)}{c-d}\left(\frac{c}{1-cz^2}-\frac{d}{1-dz^2}\right)\right]$$

where c and d are the roots of the quadratic.

$$D(u) = u^2 - (s_1s_2)^{-1}(s_0\kappa_1+s_2\kappa_2-s_1^2-s_2^2)u + 1 - \frac{\kappa_2}{s_2}$$



The “omega” expansion

Expanding the denominators

$$H_{2r}^{=}(c,d) = \sum_{m=0}^{r} B_{2r+1,2m+1}(s_1,s_2)\frac{c^{m+1}-d^{m+1}}{c-d} = \sum_{i,j=0}^{\infty} B_{2r+1,2(i+j)+1}(s_1,s_2)c^i d^j$$

where $B_{2r+1,2k+1}(s_1,s_2)$ is the "Banded Ballot polynomial"

$$B_{2r+1,2k+1}(s_1,s_2) = \mathrm{CT}\big[(\lambda\bar{\lambda})^r z^{2k}(1-z^4)\big] = \frac{k+1}{r+1}\left(\frac{s_1}{s_2}\right)^k \sum_{p=0}^{r-k}\binom{r+1}{p}\binom{r+1}{p+k+1} s_1^{2p} s_2^{2(r-}$$

Changing to "omega" variables $\omega_c = c/((s_1+cs_2)(cs_1+s_2))$

$$H_{2r}^{=}(c,d) = s_1 s_2 \frac{\omega_c Z_{2r+2}(\omega_c) - \omega_d Z_{2r+2}(\omega_d)}{c-d}$$

where, in terms of the "Banded Catalan polynomial" $C_r(s_1,s_2) = B_{2r-1,1}(s_1,s_2)$,

$$Z_{2r}(\omega) \equiv \mathrm{CT}\left[\frac{(\lambda\bar{\lambda})^{r-1}(1-z^4)}{1-\omega\lambda\bar{\lambda}}\right] = \omega_c^{-r}\left(\frac{c}{s_1 s_2} - \sum_{j=1}^{r-1} C_j(s_1,s_2)\omega_c^j\right)$$

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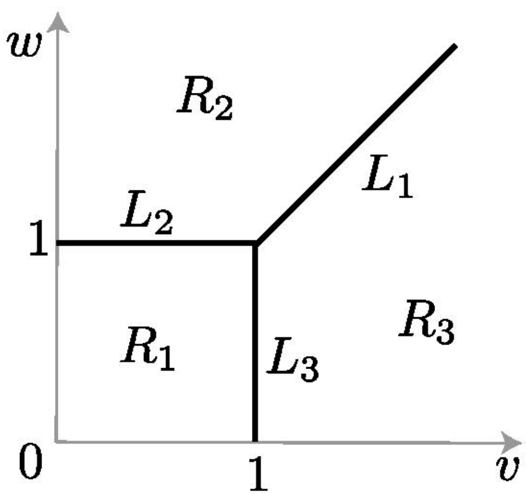
$$C_r(s_1,s_2) = \mathrm{CT}\big[(\lambda\bar{\lambda})^{r-1}(1-z^4)\big] = \sum_{i=1}^{r}\frac{1}{r}\binom{r}{i}\binom{r}{i-1} s_1^{2(i-1)} s_2^{2(r-i)} = \sum_{i=1}^{r} N_{r,i} s_1^{2(i-1)} s_2^{2(r-i)}$$

Asymptotics as r → ∞

$$Z_{2r}(\omega) = \frac{(c^2-1)\theta(c-1)}{c s_1 s_2 \omega_c^r} + \sum_{j=r}^{\infty} C_j(s_1,s_2)\omega_c^{j-r}$$

$$C_j(s_1,s_2) \simeq \frac{(s_1+s_2)^{2j+1}}{2\pi^{\frac{1}{2}}(s_1 s_2 j)^{\frac{3}{2}}} \qquad \text{as } j \to \infty$$

$$Z_{2r}(\omega_c) \sim \begin{cases} f_>(c) = \dfrac{c^2-1}{c s_1 s_2}\dfrac{1}{\omega_c^r} & c>1 \\[2ex] f_= = \dfrac{(s_1+s_2)^{2r+1}}{\pi^{\frac{1}{2}}(s_1 s_2)^{\frac{3}{2}} r^{\frac{1}{2}}} & c=1 \\[2ex] f_<(c) = \dfrac{(s_1+s_2)^{2r+1}}{2\pi^{\frac{1}{2}}(1-(s_1+s_2)^2\omega_c)(s_1 s_2)^{\frac{3}{2}} r^{\frac{3}{2}}} & c<1 \end{cases}$$



$R_2 : \dfrac{s_1 s_2 f_>(w)}{w-v}$		$L_1 : \dfrac{s_1 s_2 r (v^2-1)^2}{v^3 \omega_c^{r-1}}$
$L_2 : \dfrac{s_1 s_2 f_=}{1-v}$	$P : (s_1+s_2)^{2r}$	
$R_1 : \dfrac{s_1 s_2 (f_<(v) - f_<(w))}{v-w}$	$L_3 : \dfrac{s_1 s_2 f_=}{1-w}$	$R_3 : \dfrac{s_1 s_2 f_>(v)}{v-w}$

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C: ASEP with parallel update

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PATH TRANSFER MATRICES

To make connections with the ASEP

Dyck Paths

$$\hat{Z}_r(\kappa_1, \kappa_2) = \langle 0|(\bar{D}\bar{E})^r|0\rangle$$

$$\bar{D} = \begin{pmatrix} s_0 & s_2 & 0 & 0 & 0 & \cdots \\ 0 & s_1 & s_2 & 0 & 0 & \cdots \\ 0 & 0 & s_1 & s_2 & 0 & \cdots \\ 0 & 0 & 0 & s_1 & s_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \qquad \bar{E} = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 & 0 & \cdots \\ \kappa_2 & s_1 & 0 & 0 & \cdots & \\ 0 & s_2 & s_1 & 0 & 0 & \cdots \\ 0 & 0 & s_2 & s_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

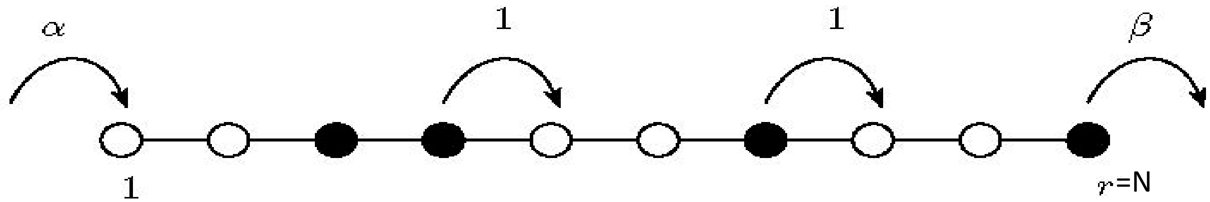
where $|0\rangle = \{1, 0, 0, \dots\}$.

Cross paths

$$H^{=}_{2r}(c, d) = \langle W_2|(D_2E_2)^r|V_2\rangle$$

$$D_2 = \begin{pmatrix} s_1 & s_2 & 0 & 0 & 0 & \cdots \\ 0 & s_1 & s_2 & 0 & 0 & \cdots \\ 0 & 0 & s_1 & s_2 & 0 & \cdots \\ 0 & 0 & 0 & s_1 & s_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \qquad E_2 = \begin{pmatrix} s_1 & 0 & 0 & 0 & 0 & \cdots \\ s_2 & s_1 & 0 & 0 & \cdots & \\ 0 & s_2 & s_1 & 0 & 0 & \cdots \\ 0 & 0 & s_2 & s_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

where $\langle W_2| = \kappa\{1, c, c^2, c^3, \dots\}$, $|V_2\rangle = \kappa\{1, d, d^2, d^3, \dots\}$ and $\kappa^2 = 1 - cd$



A: The probabilty P_p of finding p particles in the open boundary ASEP

$P_p = Z_{N,p}/Z_N$, $Z_N = \sum_{p=0}^{N} Z_{N,p}$ and the generating function for $Z_{n,p}$ is

$$G_N(x) = \sum_{p=0}^{N} Z_{N,p} x^p = \langle W_2|(xD_2 + E_2)^N|V_2\rangle = \langle W_2|(D_2 E_2(x))^N|V_2\rangle$$

where

$$E_2(x) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & \cdots \\ 1 & x & 0 & 0 & 0 & \cdots \\ 0 & 1 & x & 0 & 0 & \cdots \\ 0 & 0 & 1 & x & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

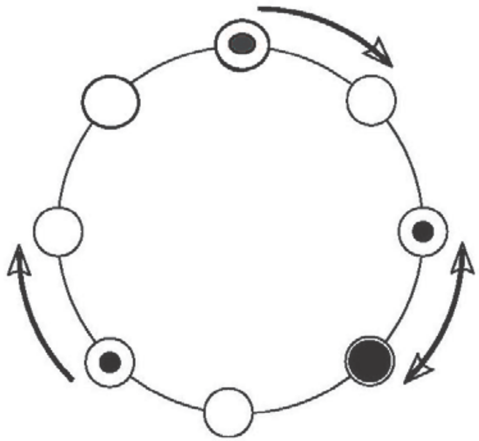
$$\bar{\alpha} = 1 + c$$
$$\bar{\beta} = 1 + d$$
$$\kappa^2 = 1 - cd$$

In the walk picture the configurations are therefore the same as for Z_N but even up steps get a weight of x. Apart from a factor $x^{(j-i)/2}$ this is equivalent to placing weight $x^{\frac{1}{2}}$ on all steps of odd height thus

$$G_N(x) = (1 - cd) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^{(j-i)/2} B_{2N+1,2(i+j)+1}(x^{\frac{1}{2}}, 1)\ c^i d^j = H_{2N}^{=}\ (cx^{-\frac{1}{2}}, dx^{\frac{1}{2}})\Big|_{\substack{s_1 = x^{\frac{1}{2}} \\ s_2 = 1}}$$

$$Z_{N,p} = \sum_{i=0}^{N-p} \sum_{j=0}^{p} \frac{i+j+1}{N+1} \binom{N+1}{p-j} \binom{N+1}{p+i+1} c^i d^j$$

Shock in the ASEP on a ring with a slow moving particle



$$10 \to 01 \quad \text{with rate} \quad 1$$
$$20 \to 02 \quad \text{with rate} \quad \alpha$$
$$12 \to 21 \quad \text{with rate} \quad \beta$$

The ring has $L+1$ sites. Mallick has shown that the velocities V_1 and V_2 of the first and second class particles are given by

$$V_2 = \frac{Y_{L-1,p} - Y_{L-1,p-1}}{Z_{L,p}} \qquad \text{and} \qquad V_1 = V_2 + \frac{L+1}{p}\frac{Y_{L-1,p-1}}{Y_{L,p}}$$

where

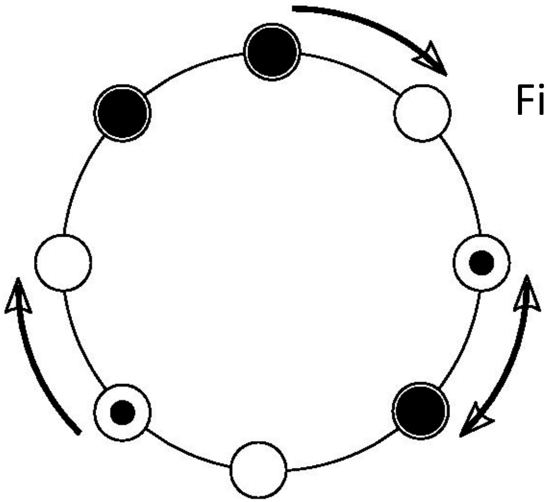
$$\sum_{p=0}^{L} Y_{L,p} x^p = Tr(A(xD_2 + E_2)^L)$$

Here A represents the second class particle. D_2 and E_2 satisfy the usual relations, $A^2 = A$, $D_2 A = \bar{\beta} A$ and $AE_2 = \bar{\alpha} A$. These relations are satisfied by $A = |V_2\rangle\langle W_2|$.

$$\sum_{p=0}^{L} Y_{L,p} x^p = Tr(|V_2\rangle\langle W_2|(xD_2 + E_2)^L) = \langle W_2|(xD_2 + E_2)^L)|V_2\rangle = G(x)$$

so $Y_{L,p} = Z_{L,p}$

B: p_1 first class and p_2 second class particles on a ring



First class = small. 12->21. Transition rates 1. N sites

A central object introduced by DJLS in their study of shocks was the grand partition function

$$G_N(x_1,x_2) = \langle 0|(x_1 D_2 + E_2 + x_2 A)^N|0\rangle$$
$$= \langle 0|(D_2 E_2(x_1,x_2))^N|0\rangle$$

where $A = |0\rangle\langle 0|$ and

$$E_2(x_1,x_2) = \begin{pmatrix} x_1+x_2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & x_1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & x_1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & x_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}$$

Rearranging the corresponding path weights to the standard s-weighted form gives $s_1 = \kappa_1 = x_1^{\frac{1}{2}}$ $s_0 = x_1^{-\frac{1}{2}}(x_1+x_2)$, $s_2 = \kappa_2 = 1$. With these values of the parameters $G_N(x_1,x_2) = \hat{Z}_N(\kappa_1,\kappa_2)$ and the roots of $D(u)$ are $c = x_2/x_1^{\frac{1}{2}}$ and $d = 0$ so

$$G_N(x_1,x_2) = H_{2N}^{=}(x_2/x_1^{\frac{1}{2}},0) = \sum_{p_2=0}^{N} x_1^{-\frac{1}{2}p_2} x_2^{p_2} B_{2N+1,2p_2+1}(x_1^{\frac{1}{2}},1)$$
$$= \sum_{p_2=0}^{N}\sum_{p_1=0}^{N-p_2} \frac{p_2+1}{N+1}\binom{N+1}{p_1}\binom{N+1}{p_1+p_2+1} x_1^{p_1} x_2^{p_2}$$

Two species of particles on a ring

Setting $\omega_d = 0$ and $\omega_c = x_2/((1+x_2)(x_1+x_2))$ in the $\omega-$form of $\hat{Z}_N(\kappa_1, \kappa_2)$

$$G_N(x_1, x_2) = \frac{s_1 s_2 \omega_c}{c} Z_{2N+2}(\omega_c) = \omega_c^{-N}\left(1 - \frac{x_1}{x_2}\sum_{j=1}^{N} C_j(s_1, s_2)\omega_c^j\right).$$

DJLS gave the following asymptotic formula, as $N \to \infty$,

$$G_N(x_1, x_2) \simeq F_N(x_1, x_2) \equiv \left(1 - \frac{x_1}{x_2^2}\right)\left(\frac{(1+x_2)(x_1+x_2)}{x_2}\right)^N$$

We find that the following formula is exact

$$G_N(x_1, x_2) = \underset{\geq}{\Omega}[F_N(x_1, x_2)].$$

where the operator $\underset{\geq}{\Omega}$ selects only the non-negative powers of x_2 in the expansion of $F_N(x_1, x_2)$.

The first few partition functions are

$$G_1(x_1, x_2) = 1 + x_1 + x_2$$
$$G_2(x_1, x_2) = 1 + 3x_1 + x_1^2 + 2x_2(1 + x_1) + x_2^2$$
$$G_3(x_1, x_2) = 1 + 6x_1 + 6x_1^2 + x_1^3 + x_2(3 + 8x_1 + 3x_1^2) + 3x_2^2(1 + x_1) + x_2^3$$

C: ASEP with parallel update

At each step any particle which can move does so with probability $p = 1-q$. The normalisation factor Z_N for the state distribution is

$$Z_N = z_N(p) + p z_{N-1}(p) \qquad (0.11)$$

where the following formula for $z_N(p)$ is equivalent to that of Evans et al

$$z_N(p) = \langle 0|(\bar{D}\bar{E})^N|0\rangle = \hat{Z}_{2N}(\kappa_1, \kappa_2)$$

Here $\bar{D}$ and $\bar{E}$ are the transfer matrices for the s and κ weighted paths with $s_0 = 1, s_1 = q^{\frac{1}{2}}, s_2 = 1, \kappa_1 = \frac{p^2}{q}(\bar{\alpha}-1)(\bar{\beta}-1), \kappa_2 = p\bar{\alpha}\bar{\beta} - \frac{p}{q}(\bar{\alpha}-1)(\bar{\beta}-1)$. $D(u)$ factorises neatly to give

$$c = \frac{p\bar{\alpha}-1}{q^{1/2}}, \qquad d = \frac{p\bar{\beta}-1}{q^{1/2}}, \qquad \omega_c = \frac{\alpha(p-\alpha)}{p^2(1-\alpha)}, \qquad \text{and} \qquad \omega_d = \frac{\beta(p-\beta)}{p^2(1-\beta)}.$$

$$z_N(p) = \hat{Z}_{2N}(\kappa_1, \kappa_2) = H_{2N}^{=}(c,d) = \sum_{m=0}^{N} b_{N,m}(q) \frac{(p\bar{\alpha}-1)^{m+1} - (p\bar{\beta}-1)^{m+1}}{p(\bar{\alpha}-\bar{\beta})}$$

where

$$b_{N,m}(q) = q^{-m/2} B_{2N+1,2m+1}(q^{\frac{1}{2}}, 1) = \frac{m+1}{N+1} \sum_{j=0}^{N} \binom{N+1}{j} \binom{N+1}{j+m+1} q^j$$

ASEP with parallel update: current

The current J_N, which is used to determine the phase diagram, is related to $z_N(p)$ by

$$J_N^{-1} = 1 + \frac{z_N(p)}{p\, z_{N-1}(p)}.$$

The asymptotic form of $z_N(p)$ as $N \to \infty$ is determined by substituting ω_c and ω_d in the general formula. There are the usual three regions

- Maximum current region $R_1 = \{\alpha > 1 - q^{\frac{1}{2}}, \beta > 1 - q^{\frac{1}{2}}\}$: $J_N = \frac{1}{2}(1 - q^{\frac{1}{2}})$
- High density region $R_2 = \{\alpha > \beta, \beta < 1 - q^{\frac{1}{2}}\}$: $J_N = \dfrac{\beta(p-\beta)}{p-\beta^2}$
- Low density region $R_3 = \{\alpha < \beta, \alpha < 1 - q^{\frac{1}{2}}\}$: $J_N = \dfrac{\alpha(p-\alpha)}{p-\alpha^2}$

ASEP with parallel update: grand partition function

The generating function for banded Catalan polynomials is

$$\bar{\Gamma}(y, s_1, s_2) \equiv \sum_{r=0}^{\infty} C_r(s_1, s_2) y^r$$

and with $\bar{\Gamma} \equiv \bar{\Gamma}(y, q^{\frac{1}{2}}, 1)$ the g.p.f. for the parallel update ASEP is

$$\mathcal{Z}_p(y) \equiv \sum_{N=0}^{\infty} Z_N y^N = (1 + py)\Xi_{\|}(y, p)$$

where

$$\Xi_{\|}(y, p) \equiv \sum_{N=0}^{\infty} z_N(p) y^N = \frac{(\bar{\Gamma} - 1)(1 - qy\bar{\Gamma})^2}{y(1 - p(\bar{\alpha} - 1)\bar{\Gamma}\, y)(1 - p(\bar{\beta} - 1)\bar{\Gamma}\, y))}$$

in agreement with Blythe et al equation (44). We have four such formulae, a second one is

$$\Xi_{\|}(y, p) = \frac{(\bar{\Gamma} - 1)(1 - (p + q\bar{\Gamma})y)}{(1 - p\bar{\alpha}(p + q\bar{\Gamma})y)(1 - p\bar{\beta}(p + q\bar{\Gamma})y)}$$

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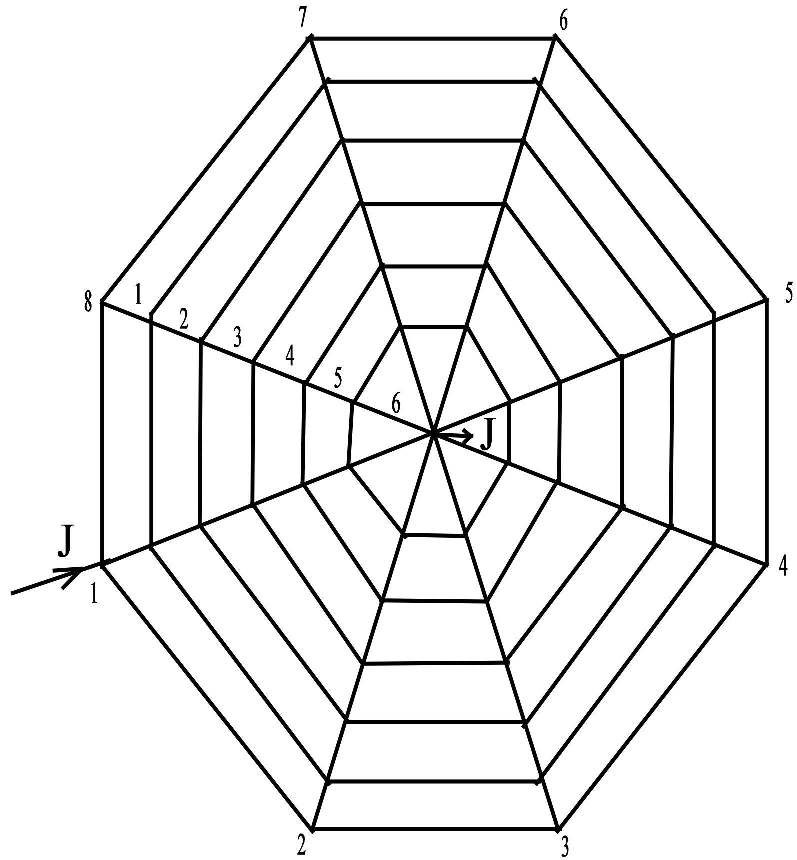

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